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**Topics on Hyperbolic Function Theory in Cl_{n+1,0}**
Topics on Hyperbolic Function Theory in $\mathcal{C}l_{n+1,0}$

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Abstract

Left hypergenic functions are Clifford algebra $\mathcal{C}l_{n+1,0}$-valued functions satisfying the equation $Df - \frac{1}{n}Q_0 f = 0$ where $D$ is the Dirac operator and $Q_0$ is the projection-type mapping $\mathcal{C}l_{n+1,0} \to \mathcal{C}l_{n,0}$ given by composition. Similarly we define right hypergenic functions taking right actions in the previous definition. We consider some fundamental and local properties of hypergenic functions. We prove our version of the Borel-Pompeiu formula and a new version of Cauchy's theorem. We shall consider hypergenic Clifford algebra-values multivector functions and prove that in that case left- and right hypergenicity coincide.

1 Preliminaries

Let $\{e_0, ..., e_n\}$ be the standard basis in $\mathbb{R}^{n+1}$. The Clifford algebra $\mathcal{C}l_{n+1}$ is the free associative algebra with unit generated by standard basis vectors together with the defining relations

$e_i e_j + e_j e_i = 2\delta_{ij}$

for each $i, j = 0, ..., n$. The Clifford algebra $\mathcal{C}l_{n+1}$ has the dimension $2^{n+1}$ and the canonical basis is given by $e_A = e_{a_1} \cdots e_{a_k}$ where $A = \{a_1, ..., a_k\} \subset N = \{0, ..., n\}$ and $a_1 < \cdots < a_k$. Especially $e_0 = 1$ and $e_{\{j\}} = e_j$. The space of $k$-vectors is defined by $\mathcal{C}l^k_{n+1} = \text{span}\{e_A : |A| = k\}$. Any $a \in \mathcal{C}l_{n+1}$ admit the multivector decomposition:

$a = [a]_0 + [a]_1 + \cdots + [a]_{n+1}$

with $[a]_k \in \mathcal{C}l^k_{n+1}$. The space of 0-vectors is identified with $\mathbb{R}$ and the set of 1-vectors is identified with $\mathbb{R}^{n+1}$.

Assume that $\Omega$ is an open subset of $\mathbb{R}^{n+1}$. In the canonical basis every function $f : \Omega \to \mathcal{C}l_{n}$ admit the representation

$f = \sum_A e_A f_A$.

A function $f$ is called differentiable in $\Omega$ if $f_A$ is differentiable in $\Omega$ for each $A$. If $f$ is differentiable we define the left Dirac operator by

$D_{\ell}f = \sum_{k=0}^n e_k \frac{\partial f}{\partial x_k}$

and the right Dirac operator by

$D_{r}f = \sum_{k=0}^n \frac{\partial f}{\partial x_k} e_k$.
where derivatives operate componentwise. Denoting $\mathcal{C}l_n$ the Clifford algebra generated by $\{e_1, \ldots, e_n\}$. We may represent the Clifford algebra $\mathcal{C}l_n$ as the direct sum

$$\mathcal{C}l_{n+1} = \mathcal{C}l_n \oplus e_0 \mathcal{C}l_n.$$ 

Let $\pi_1$ and $\pi_2$ be the corresponding projections i.e., $\pi_1(a + e_0 b) = a$ and $\pi_2(a + e_0 b) = e_0 b$ and let $\mu : \mathcal{C}l_{n+1} \to \mathcal{C}l_{n+1}$ be the involution $\mu(a) = e_0 a$. Using the previous mappings we define

$$P_0 := \pi_1 \text{ and } Q_0 := \mu \circ \pi_2.$$

Let $\Omega$ be an open subset of $\mathbb{R}^{n+1}$ contained in the upper half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^{n+1} \cap \{x_0 > 0\}$. We define the left- and right-modified Dirac operator on the open set $\Omega$ using the previous mappings by

$$H^L_k f = D_0 f - \frac{k}{x_0} Q_0 f,$$

$$H^R_k f = D_1 f - \frac{k}{x_0} Q_0' f$$

where $k$ is an arbitrary real number. We shall also use abbreviated notations $H^L := H^L_{(n-1)}$ and $H^R := H^R_{(n-1)}$ for index $k = n - 1$. Null solutions of the previous operators are called left- and right-hypergenic functions.

As a technical tool we will need $P_0$ and $Q_0$-parts of the operators $H^L_k$ and $H^R_k$, represented in the next lemma.

**Lemma 1.1** Let $f : \Omega \to \mathcal{C}l_{n+1}$ be a smooth function. Then

(a) $P_0(\mathcal{H}^L_k f) = D_0 P_0 f + \frac{\partial Q_0 f}{\partial x_0} - \frac{k}{x_0} Q_0 f$,

(b) $Q_0(\mathcal{H}^L_k f) = \frac{\partial P_0 f}{\partial x_0} - D_0 Q_0 f$,

(c) $P_0((\mathcal{H}^L_k f)^2 f) = \Delta P_0 f - \frac{k}{x_0} \frac{\partial P_0 f}{\partial x_0}$,

(d) $Q_0((\mathcal{H}^L_k f)^2 f) = \Delta Q_0 f - \frac{k}{x_0} \frac{\partial Q_0 f}{\partial x_0} + \frac{k}{x_0^2} Q_0 f$,

where $D_1 = e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}$.

**Proof.** We will prove only (a) and (b) since the proof for (c) and (d) can be found in the proof of the Theorem 1.8 of [5]. Assume $f = P_0 f + e_0 Q_0 f$ is a smooth function. Since $D_1 = e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}$ we obtain

$$D(P_0 f) = e_0 \frac{\partial P_0 f}{\partial x_0} + D_1 P_0 f$$

and

$$D(e_0 Q_0 f) = \frac{\partial Q_0 f}{\partial x_0} - e_0 D_1 Q_0 f.$$ 

Previous observations implies that

$$H^L_k f = H^L_k (P_0 f + e_0 Q_0 f)$$

$$= D(P_0 f) + D(e_0 Q_0 f) - \frac{k}{x_0} Q_0 f$$

$$= \left\{D_1 P_0 f + \frac{\partial Q_0 f}{\partial x_0} - \frac{k}{x_0} Q_0 f\right\} + e_0 \left\{\frac{\partial P_0 f}{\partial x_0} - D_1 Q_0 f\right\}.$$ 

Applying $P_0$ and $Q_0$ we obtain the result. \[\square\]
2 Local Properties of Hypergenic Functions

In this section we consider local properties of hypergenic functions. If $\Omega$ is a domain in $\mathbb{R}^{n+1}$ the operator

$$\Delta_{LB} = x_0^2 \left( \Delta g - \frac{k}{x_0} \frac{\partial g}{\partial x_0} \right)$$

is the Laplace-Beltrami operator for $g \in C^2(\Omega, \mathbb{C}^{n+1})$ with respect to the Riemannian metric

$$ds^2 = dx_0^2 + \ldots + dx_n^2.$$

In [5] we obtain the following theorem:

**Theorem 2.1** Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain and $f \in C^2(\Omega, \mathbb{C}^{n+1})$ be a $k$-hypergenic. The function $P_0 f$ is a solution of the Laplace-Beltrami equation i.e. $\Delta_{LB} P_0 f = 0$ and $Q_0 f$ is a solution of the eigenvalue problem $\Delta_{LB} Q_0 f = -k Q_0 f$.

We see, that the $Q_0$-part of a hypergenic function is an eigenfunction of the Laplace-Beltrami operator.

**Theorem 2.2** Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain. A smooth function $f : \Omega \to \mathbb{C}^{n+1}$ is hypergenic if and only if for any $a \in \Omega$ there exist $r > 0$ satisfying $B_r(a) \subset \Omega$ and a map $g : B_r(a) \to \mathbb{C}^{n,0}$ satisfying

$$f = Dg$$

and

$$\Delta_{LB} g = 0.$$

**Proof.** Assume first that the radius and the function $g$ exist. Then in the neighborhood of $a$

$$H^s f = \Delta g - \frac{k}{x_0} Q_0(Dg) = \Delta g - \frac{k}{x_0} \frac{\partial g}{\partial x_0} = \frac{1}{x_0} \Delta_{LB} g = 0.$$

Assume that $f : \Omega \to \mathbb{C}^{n+1}$ is hypergenic. Let us denote that $x = (x_0, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^n$ and $\Delta_1 := D_1 D_{\tilde{x}}$. Assume that $s_A$ is a solution of the Poisson problem

$$\Delta_1 s_A(\tilde{x}) = (P_0 f)_A(a_0, \tilde{x})$$

in $B_r(a) \cap \mathbb{R}^n$ and let $s = \sum_{A \subset \{1, \ldots, n\}} s_A$. The solution of the Poisson problem exits, for example see [1]. Then we define

$$g(x) = \int_{a_0}^{x_0} Q_0 f(t, \tilde{x}) s dt + D_1 s(\tilde{x})$$

and obtain

$$Dg(x) = c_0 Q_0 f(x) + \int_{a_0}^{x_0} D_1 Q_0 f(t, \tilde{x}) dt + \Delta_1 s(\tilde{x})$$
Using Lemma 1.1 we obtain that \( D_1 Q_0 f = \frac{\partial P_0 f}{\partial x_0} \) and
\[
Dg(x) = e_0 Q_0 f(x) + \int_{a_0}^{x_0} \frac{\partial P_0 f}{\partial x_0}(t, \tilde{x}) dt + P_0 f(a_0, \tilde{x}) = f(x).
\]
Similarly we see that
\[
0 = Df - \frac{k}{x_0} Q_0 f = \Delta g - \frac{k}{x_0} \frac{\partial g}{\partial x_0},
\]
which completes the proof. 

3 On Euler Operator in the Class of Hypergenic Functions

The Euler operator is defined by
\[
E = \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}.
\]
We see that it is a scalar operator and measures the degree of homogeneity of a function.

**Lemma 3.1** If \( D \) is the Dirac operator and \( E \) is the Euler operator we have:
\[
DE = D + ED.
\]
**Proof.** Let \( f : \Omega \to \mathcal{C}\ell_{n+1} \) be a differentiable function. Then
\[
DEf = \sum_{j,i} e_j \frac{\partial}{\partial x_j} \left( x_i \frac{\partial f}{\partial x_i} \right)
\]
\[
= \sum_{j,i} e_j \delta_{i,j} \frac{\partial f}{\partial x_j} + \sum_{j,i} e_j x_i \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j}
\]
\[
= \sum_{j} e_j \frac{\partial f}{\partial x_j} + \sum_{j,i} x_i \frac{\partial}{\partial x_i} e_j \frac{\partial f}{\partial x_j}
\]
\[
= Df + Edf,
\]
and we obtain the result. 

The similar result holds in the class of hypergenic functions:

**Theorem 3.2** Let \( f : \Omega \to \mathcal{C}\ell_{n+1} \) be a smooth function. If the function \( f \) is \( k \)-hypergenic then \( Ef \) is \( k \)-hypergenic.

**Proof.** We assume that \( Df - \frac{k}{x_0} Q_0 f = 0 \). Applying \( E \) to \( \frac{k}{x_0} Q_0 f \) we obtain
\[
\frac{k}{x_0} Q_0 Ef = \frac{k}{x_0} Q_0 f + E\left( \frac{k}{x_0} Q_0 f \right).
\]
Using previous lemma we obtain

\[ H_kEf = DEF - \frac{k}{x_0}Q_0Ef \]

\[ = Df + Edf - \frac{k}{x_0}Q_0f - E\left(\frac{k}{x_0}Q_0f\right) \]

\[ = Df - \frac{k}{x_0}Q_0f + E\left(Df - \frac{k}{x_0}Q_0f\right) = 0. \]

The proof is complete. \[\blacksquare\]

If \( f \) is a hypergenic function the previous theorem gives us a method how to construct more hypergenic functions:

The above theorem implies that for every \( k \)-hypergenic function \( f : \Omega \to \mathcal{C}^{n+1,0} \) there exists the sequence of hypergenic functions defined by

\[ E^m f := E \cdots E f \]

for each \( m \in \mathbb{N} \) and \( E^0 f = f \). This sequence \( \{E^m f : m = 0, 1, \ldots\} \) is called the homogenized sequence of \( f \).

As an example we consider monomials

\[ X_i = \sum_{j=0}^{n} (-1)^{i_j}x_je_j \]

where \( i = 1, \ldots, n \). Since \( H^i X_i = 0 \) the monomials \( X_i \) are hypergenic. Moreover

\[ EX_i = X_i \]

for each \( i = 1, \ldots, n \) and hence the homogenized sequence of \( X_i \) is just \( \{X_i\} \).

### 4 Further Representation Results

In this section we will consider the improved version of the Cauchy’s formula represented in \([5]\). The second aim is to prove the Borel-Pompeiu formula in the language of \( H \)-operators.

First we recall briefly some preliminaries from integration theory. Let \( M \) be a be a \( k \)-dimensional manifold-with-boundary in \( \mathbb{R}^{n+1}_+ \), see e.g. \([12]\). The boundary of \( M \) is denoted by \( \partial M \). If moreover

\[ \Lambda^* M = \bigoplus_{p=0}^{n+1} \Lambda^p M \]

is the exterior algebra over \( \mathbb{R}^{n+1}_+ \) with basis \( \{dx_0, dx_1, ..., dx_n\} \) we then construct the bundle \( \mathcal{C}^{n+1}_+ \otimes_{\mathbb{R}} \Lambda^k M \). If \( \omega(x) \) is a section of the previous bundle over \( x \in M \) it is of the form

\[ \omega(x) = \sum_{A,B} \omega_{A,B}(x)e_A dx_B \]
where \( B = \{ b_1, \ldots, b_k \} \subset N = \{ 1, \ldots, n \} \) and \( dx_B = dx_{b_1} \wedge \cdots \wedge dx_{b_k} \). The meaning of the symbol \( e_A dx_{B} \) is clear. Let furthermore \( M \) be an oriented \( k \)-dimensional manifold-with-boundary in \( \mathbb{R}^{n+1} \), then we define

\[
\int_{M} \omega(x) = \sum_{A,B} e_A \int_{M} \omega_{A,B}(x) dx_B.
\]

In this paper \( M \) will be \((n+1)\)-dimensional or \( n \)-dimensional i.e. the boundary of \( M \). The \((n+1)\)-form

\[
dV = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n
\]
on \( M \) is called the (Riemannian) volume element. In surface integrals we shall often use the \( n \)-form

\[
d\sigma = \sum_{i=0}^{n} (-1)^i e_i \hat{d} x_i
\]

where

\[
\hat{d} x_i = \hat{dx}_0 \wedge \hat{dx}_1 \wedge \cdots \wedge \hat{dx}_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n
\]

for each \( i = 0, 1, \ldots, n \). The exterior derivative \( d \) for Clifford algebra valued differential forms is defined componentwise, i.e., if

\[
\omega = \sum_{A,B} \omega_{A} e_B
\]
is a \( k \)-form then

\[
d\omega = \sum_{A,B} d\omega_{A} e_B.
\]

Applying the classical Stokes theorem (see [12]) it is easy to prove that:

**Theorem 4.1 (Stokes)** Let \( \omega \) be a \( \mathcal{C}_{n+1} \)-valued \( k \)-form in the oriented \( k \)-dimensional manifold-with-boundary \( M \). Then

\[
\int_{M} d\omega = \int_{\partial M} \omega.
\]

Let us denote the interior of the subset \( U \subset \mathbb{R}^n \) by \( U^\circ \). We recall the Cauchy’s formula, represented in [5].

**Theorem 4.2** Assume that \( \Omega \) is an open subset of \( \mathbb{R}^{n+1}_+ \) and \( f : \Omega \to \mathcal{C}_{n+1} \) is hypergenic. Let \( M \subset \Omega \) be an oriented \((n+1)\)-dimensional manifold-with-boundary. Then

\[
f(y) = \frac{2^{n-1} y_{n+1}^{n-1}}{\omega_{n+1}} \int_{\partial M} \frac{(x-y)^{-1}d\sigma(x)f(x) - (\hat{x} - y)^{-1}d\sigma(\hat{x})\hat{f}(x)}{|x-y|^{n-1}|x-\hat{y}|^{n-1}}
\]

for each \( y \in M^\circ \).

If we decompose

\[
d\sigma(x)f(x) = P_0(d\sigma(x)f(x)) + e_0Q_0(d\sigma(x)f(x))
\]

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we obtain
\[(x - y)^{-1}d\sigma(x)f(x) - (\tilde{x} - y)^{-1}\tilde{d}\sigma(x)\tilde{f}(x)\]
\[= ((x - y)^{-1} - (\tilde{x} - y)^{-1})P_0(d\sigma(x)f(x)) + ((x - y)^{-1} + (\tilde{x} - y)^{-1})e_0Q_0(d\sigma(x)f(x)). \tag{1}\]

If we define a new kernel
\[\Lambda(x, y) := \frac{(x - y)^{-1} - (\tilde{x} - y)^{-1}}{|x - y|^{n-1}|x - y|^{n-1}}e_0x_0 = \Lambda(x, y)(y - Px),\]
we get

**Lemma 4.3**
\[\frac{(x - y)^{-1} + (\tilde{x} - y)^{-1}}{|x - y|^{n-1}|x - y|^{n-1}}e_0x_0 = \Lambda(x, y)(y - Px).\]

Using the identity (1) and the above lemma we obtain that
\[\frac{(x - y)^{-1}d\sigma(x)f(x) - (\tilde{x} - y)^{-1}\tilde{d}\sigma(x)\tilde{f}(x)}{|x - y|^{n-1}|x - y|^{n-1}}\]
\[= \Lambda(x, y)P_0(d\sigma(x)f(x)) + \Lambda(x, y)y - Px\frac{Q_0(d\sigma(x)f(x))}{x_0}.\]

Hence we obtain an other version of the Cauchy’s formula:

**Theorem 4.4** Assume that \(\Omega\) is an open subset of \(\mathbb{R}^{n+1}_{+}\) and \(f : \Omega \to \mathbb{C}_{n+1}\) is hypergenic. Let \(M \subset \Omega\) be an oriented \((n+1)\)-dimensional manifold-with-boundary. Then
\[f(y) = \frac{2^{n-1}}{\omega_{n+1}}\int_{\partial M} \Lambda(x, y)\left(P_0(d\sigma(x)f(x)) + \frac{y - Px}{x_0}Q_0(d\sigma(x)f(x))\right).\]
for each \(y \in M^\circ\).

**Theorem 4.5 (Borel-Pompeiu Formula)** Assume that \(\Omega\) is an open subset of \(\mathbb{R}^{n+1}_{+}\) and \(f : \Omega \to \mathbb{C}_{n+1}\) is differentiable. Let \(M \subset \Omega\) be an oriented \((n+1)\)-dimensional manifold-with-boundary. Then
\[f(y) = \frac{2^{n-1}}{\omega_{n+1}}\int_{\partial M} \left\{\frac{1}{x_0^{n-1}}P_0(p(x, y)d\sigma(x)f(x)) + e_0Q_0(q(x, y)d\sigma(x)f(x))\right\}\]
\[+ \frac{2^{n-1}}{\omega_{n+1}}\int_{M} \left\{\frac{1}{x_0^{n-1}}P_0(p(x, y)H^d(f(x)) + e_0Q_0(q(x, y)H^f(f(x))\right\}dV.\]
for each \(y \in M^\circ\) where \(p(x, y) := x_0^{n-1}(x - y)^{-1}(x - y)^{-1}\) and \(q(x, y) := \frac{(x - y)^{-1} + (\tilde{x} - y)^{-1}}{|x - y|^{n-1}|x - y|^{n-1}}\) are hypergenic with respect to \(x\).

**Remark 4.6** In Borel-Pompeiu formula hypergenicity has no role. Assuming that \(f\) is hypergenic we obtain original version of Cauchy’s formula.

In order to prove Borel-Pompeiu’s theorem, let us first state a few basic lemmata. All proofs we refer [3].
Lemma 4.7 The function $x \mapsto p(x, y)$ is right $(n-1)$-hypergenic and the function $x \mapsto q(x, y)$ is right $-(n-1)$-hypergenic on $\mathbb{R}^{n+1}$, $\{y, \tilde{y}\}$ for each $y \in \mathbb{R}^{n+1}$.

Lemma 4.8 Assume that $\Omega$ is an open set of $\mathbb{R}^{n+1}$ and $f, g : \Omega \to C^\ell_{n+1}$ are differentiable functions. Let $M \subset \Omega$ be an oriented $(n+1)$-dimensional manifold-with-boundary. If $g$ is a right hypergenic then
\[ \int_{\partial M} \frac{1}{x_0} P_0(g \sigma f) = \int_{M} \frac{1}{x_0^2} P_0(g H^f) dV \]
and if $g$ is a right hypergenic then
\[ \int_{\partial M} Q_0(g \sigma f) = \int_{M} Q_0(g H^f) dV. \]

Proof. This is a corollary of Lemma 2.6 and Lemma 2.9 in [5].

Lemma 4.9 If $f : \Omega \to C^\ell_{n+1, 0}$ is continuous and $B_r(y) \subset \Omega \subset \mathbb{R}^{n+1}$ then
\[ \frac{2^{n-1} y_0^{n-1}}{\omega_{n+1}} \int_{\partial B_r(y)} \frac{1}{x_0^n} P_0(p(x, y) \sigma(x)f(x)) \to P_0 f(y) \]
and
\[ \frac{2^{n-1} y_0^{n-1}}{\omega_{n+1}} \int_{\partial B_r(y)} Q_0(q(x, y) \sigma(x)f(x)) \to Q_0 f(y) \]
as $r \to 0$.

Proof. See the proofs of Theorem 2.4 and 2.7 in [5].

Proof of Theorem 4.5. Since $M^o$ is an open set for each $y \in M^o$ there exists $r > 0$ such that $B_r(y) \subset M^o$. We denote $M_r(y) := M \setminus B_r(y)$. Then
\[
\begin{align*}
\int_{\partial B_r(y)} \frac{1}{x_0^n} P_0(p(x, y) \sigma(x)f(x)) + e_0 \int_{\partial M_r(y)} Q_0(q(x, y) \sigma(x)f(x)) \\
= & \int_{\partial M_r(y)} \frac{1}{x_0^n} P_0(p(x, y) \sigma(x)f(x)) + e_0 \int_{\partial M_r(y)} Q_0(q(x, y) \sigma(x)f(x)) \\
& + \int_{\partial B_r(y)} \frac{1}{x_0^n} P_0(p(x, y) \sigma(x)f(x)) + e_0 \int_{\partial B_r(y)} Q_0(q(x, y) \sigma(x)f(x)).
\end{align*}
\]
Using Lemma 4.9 we obtain
\[
\int_{\partial B_r(y)} \frac{1}{x_0^n} P_0(p(x, y) \sigma(x)f(x)) + e_0 \int_{\partial M_r(y)} Q_0(q(x, y) \sigma(x)f(x)) \to \frac{\omega_{n+1}}{2^{n-1} y_0^{n-1}} f(y)
\]
as $r \to 0$. Using Lemma 4.7 and 4.8 we obtain
\[
\int_{\partial M_r(y)} \frac{1}{x_0^{n-1}} P_0(p(x, y) \sigma(x)f(x)) = \int_{M_r(y)} \frac{1}{x_0^{n-1}} P_0(p(x, y) H^f(x)) dx
\]
and
\[
\int_{\partial M_r(y)} Q_0(q(x, y) \sigma(x)f(x)) = \int_{M_r(y)} Q_0(q(x, y) H^f(x)) dx.
\]
Since $\int_{M_r(y)} \to \int_M$ as $r \to 0$, we obtain the result just taking $r \to 0$ in (2).

Remark 4.10 In the proof there are no problems with integration since the Stokes theorem holds also on the manifold with corners, see [8].
5 Multivector Calculus

Lastly in [5] we study some connections between \(Q_k\)-operator, \(H_{\ell}\)-operators and left- and right contraction -operators. We recall some notions and properties.

Let \(x \in \mathcal{C}_{n+1}^1\) be a vector and \(u \in \mathcal{C}_{n+1}\) be an arbitrary Clifford number. The left contraction is defined by

\[
x \cdot u = \frac{1}{2}(xu - u'x),
\]

and the right contraction by

\[
u \cdot x = \frac{1}{2}(ux - xu').
\]

Marcel Riesz has introduced in 1958 the exterior product in Clifford algebras. The exterior product is defined by

\[
x \wedge u = \frac{1}{2}(xu + u'x)\]

(4)

Adding (3) and (4) together we obtain the Clifford product

\[
xu = x \cdot u + u \wedge x.
\]

Moreover if \(u\) is an \(r\)-vector we have

\[
x \cdot u = (-1)^{r-1}u \cdot x
\]

for each vector \(x\). Historical remarks and more comprehensive introduction to the contraction products and the exterior product, see [11].

**Remark 5.1** If \(u, x \in \mathcal{C}_{n+1}^1\), then

\[
x \cdot u = u \cdot x = (x, u)
\]

where \((\cdot, \cdot)\) is the usual Euclidean inner product.

As in previous remark we noticed that for each \(x \in \mathcal{C}_{n+1}^1\) the mapping \(u \mapsto x \cdot u\) or \(u \mapsto u \cdot x\) is a mapping \(\mathcal{C}_{n+1}^k \to \mathcal{C}_{n+1}^{k-1}\) for each \(k\). It is easy to notice that this phenomenon should be true in generally. To see that, let us denote \(L_x u := x \cdot u\), \(R_x u := u \cdot x\) and \(W_x u = x \wedge u\) where all mappings are from \(\mathcal{C}_{n+1}^k\) into \(\mathcal{C}_{n+1}^{k-1}\). Hence it’s obvious that

\[
L_x, R_x : \mathcal{C}_{n+1}^k \to \mathcal{C}_{n+1}^{k-1} \quad \text{and} \quad W_x : \mathcal{C}_{n+1}^k \to \mathcal{C}_{n+1}^{k+1}
\]

for each \(1 \leq k \leq n - 1\).

Let \(f : \Omega \to \mathcal{C}_{n+1}\) be a \(C^1\)-function where \(\Omega\) is an open set. As in [5] we deduce that hyperbolic Dirac operators are represented using the previous operators in the form

\[
H_k f = D_k f - \frac{k}{x_0} e_0 f, \quad H_k^* f = D_k f - \frac{k}{x_0} f e_0,
\]

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on $\Omega$.

We interpret the Dirac operator $D_\ell$ as a vector. As in [7] we also interpret $D_\ell f$ as a product of a vector and a Clifford number i.e.

$$D_\ell f = D_\ell \cdot f + D_\ell \wedge f.$$ 

Since $D_\ell f = L_D f$ and $D \wedge f = W_D f$ we obtain that

$$D_\ell = L_D + W_D.$$ 

That allow us to ask: Is there any corresponding splitting for $H_\ell$-operator? An answer is simple. Using the above formulas we compute:

$$H_\ell^f f = D_\ell f - \frac{k}{x^0} e_0 \cdot f$$

$$= D_\ell \cdot f + D_\ell \wedge f - \frac{k}{x^0} e_0 \cdot f$$

$$= (D_\ell - \frac{k}{x^0} e_0) \cdot f + D_\ell \wedge f.$$ 

Hence

$$H_\ell^f = L_{(D_\ell - \frac{k}{x^0} e_0)} + W_D.$$ 

Similarly as in the previous discussion we obtain that

$$H_\ell^f = R_{(D_\ell - \frac{n-1}{x^0} e_0)} + W_D.$$ 

Recall that $H^\ell := H^\ell_{(n-1)}$, $H^r := H^r_{(n-1)}$, $H^r_\ell := R_{(D_\ell - \frac{k}{x^0} e_0)}$, $H^r_\ell := R_{(D_\ell - \frac{n-1}{x^0} e_0)}$ and $H^r_\ell = H^r_\ell + E_D$. Thus in the class of $k$-vector valued functions

$$H^\ell = H^\ell_\ell + H^\ell_\ell$$

and

$$H^r = H^r_\ell + H^r_\ell.$$ 

**Theorem 5.2** Let $f \in C^1(\Omega, C^{p+1})$ be a $p$-vector, for $p = 1, \ldots, n$. Then

$$H^\ell_\ell f = 0,$$

$$H^r_\ell f = 0$$

if and only if

$$H^\ell f = 0.$$ 

**Proof.** Assume that

$$H^\ell f = (D_\ell - \frac{n-1}{x^0} e_0) \cdot f + D_\ell \wedge f = 0.$$ 

Since

$$[H^\ell f]_{p-1} = (D_\ell - \frac{n-1}{x^0} e_0) \cdot f = 0$$

and

$$[H^\ell f]_{p+1} = D_\ell \wedge f = 0$$

so we obtain the result. ■
If the function \( f \in C^1(\Omega, \mathcal{C}^{\ell}_{n+1}) \) is a product of functions i.e., there exist a vector valued function \( g \) and a \( p \)-vector \( h \) satisfying \( f = gh \). That allow us to split
\[
f = f_+ + f_-
\]
where \( f_+ = g \wedge h \) and \( f = g \cdot h \). Using previous theorem we obtain:

**Corollary 5.3** Let \( f \in C^1(\Omega, \mathcal{C}^{\ell}_{n+1}) \) be a product \( f = gh \) where \( g \in C^1(\Omega, \mathcal{C}^{\ell}_{1,n+1}) \) and \( h \in C^1(\Omega, \mathcal{C}^{\ell}_{p,n+1}) \). If
\[
H^{\ell} f_+ = 0, \\
H^{\ell} f_+ = 0, \\
H^{\ell} f_- = 0, \\
H^{\ell} f_- = 0,
\]
where \( f_+ = g \wedge h \) and \( f_- = g \cdot h \), then \( f \) is a hypergenic function.

In the case of \( k \)-vector valued functions it is comfortable to represent \( P_0 \) and \( Q_0 \)-operators using the Riesz’s products:

**Proposition 5.4** If \( u \in \mathcal{C}^{\ell}_{n+1} \) then
\[
\begin{align*}
1. & \quad P_0 u = e_0 \rho(e_0 \wedge u) \quad \text{and} \quad Q_0 u = e_0 \rho u, \\
2. & \quad P'_0 u = (e_0 \wedge u)_\cdot e_0 \quad \text{and} \quad Q'_0 u = u \cdot e_0.
\end{align*}
\]

**Proof**

1. The proof for \( Qu = e_0 \rho u \), see [5].

\[
e_0 u + u' e_0 = e_0 Pu + Qu + P'u e_0 - e_0 Q'u e_0 = 2e_0 Pu
\]

we obtain
\[
Pu = e_0 \frac{1}{2}(e_0 u + u' e_0) = e_0 (e_0 \wedge u) = e_0 \rho(e_0 \wedge u)
\]

since \( e_0 \wedge e_0 \wedge u = 0 \).

2. Recall \( P'_0 u = (P_0 u)' \) and \( Q'_0 u = (Q_0 u)' \). Since
\[
(e_0 \cdot v)' = e_0 v' - e_0 v = v \cdot e_0
\]

for each \( v \in \mathcal{C}^{\ell}_{n+1,0} \) we obtain that
\[
P'_0 u = (e_0 \cdot (e_0 \wedge u))' = (e_0 \wedge u)_\cdot e_0
\]

and \( Q'_0 u = u \cdot e_0 \). \[\blacksquare\]

The Theorem 5.2 is formulated and proved only for the left hypergenic functions. In the next theorem we shall show that this is not any shortcoming of the theory.

**Theorem 5.5** For any \( k \)-vector valued function the notions of left hypergenicity and right hypergenicity coincide.
Proof. Assume that \( f \) is a \( k \)-vector valued function. Then

\[
[H^e_{k-1}] = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} - \frac{n-1}{x_0} e_0 f
\]

\[
= (-1)^k \left( \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} e_i - f e_0 \frac{n-1}{x_0} \right)
\]

\[
= (-1)^k [H^r_{k-1}].
\]

Since \([H^e_{k-1}] = [H^r_{k-1}]\) we obtain the result. □

References


